EQUILONG INVARIANTS AND CONVERGENCE PROOFS.

BY PROFESSOR EDWARD KASNER.

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THE writer has studied the invariants of a pair of analytic curves under the equilong group with the main object of throwing light on the corresponding question in the more important conformal geometry.* The two theories present many analogies, but are not connected by a strict principle of duality. The number of invariants and their orders turn out to be the the same, though the results have to be calculated independently.

In some questions, however, the two theories differ essentially, not only in the methods to be employed, but also in the results obtained. This is true, in particular, with regard to the convergence of the power series entering into the formal calculations. This question was left unsettled in the paper cited.

The principal object of the present paper is to complete the equilong theory by showing that the series in question are always convergent. It thus follows that the equality of the absolute invariants is a sufficient as well as a necessary condition for the equivalence of two pairs of curves. The method used is to reduce the question to one in differential equations† and then to apply certain existence theorems, for solutions at a singular point, due to Briot and Bouquet.

1. *Calculation of the Invariants.*

The equilong group of the plane consists of all contact transformations which convert straight lines into straight lines in such a way that the distance δ between the points of contact of any two curves on a common tangent remains in-

* See Conformal Geometry, Proceedings of the Fifth International Congress, Cambridge (1912), vol. 2, pp. 81–87.

† Such a reduction is impossible in the conformal theory. We have instead a functional equation, and in some cases, as Dr. Pfeiffer has recently shown, the formal solution is actually divergent. In the language of the paper cited above, invariant relations of infinite order are required in conformal equivalence, but not in equilong equivalence.

variant. If we use Hessian line coordinates u, v (where v is the perpendicular distance from the origin and u is the angle which the perpendicular makes with the initial line), the group may be written, in the notation of dual numbers,

$$(1) \qquad U + jV = \text{Function } (u + jv), \quad \text{where} \quad j^2 = 0,$$

or, in separated form,

$$(2) \qquad U = \varphi(u), \quad V = v\varphi'(u) + \psi(u),$$

where φ and ψ are arbitrary analytic functions.* If we change the sign of v, we obtain the improper equilong transformations which preserve the magnitude of δ but reverse its sense.

A single regular analytic curve has no invariants: it can always be reduced to the normal form $v = 0$, that is, the origin (considered as an envelope of lines).

Let us now consider two curves having a common tangent; this tangent we may assume to be the line $u = 0$, $v = 0$. One of the curves we may assume reduced to $v = 0$. The other is defined say by $v = f(u)$, where f is any power series without a constant term. In the second plane let the two curves be written in the form $V = 0$, and $V = F(U)$.

We then have to consider the subgroup of (2) which converts the point $v = 0$ into itself, and the line $u = 0, v = 0$ into itself. This is

$$(3) \qquad U = \varphi(u), \quad V = v\varphi'(u),$$

where $\varphi(u)$ is any power series beginning with the first power of u.

In terms of power series our problem is now as follows: When will the curves

$$(4) \qquad v = \alpha_1 u + \alpha_2 u^2 + \cdots,$$

$$(5) \qquad V = A_1 U + A_2 U^2 + \cdots$$

be equivalent under a transformation of the form

$$(6) \qquad \begin{aligned} U &= a_1 u + a_2 u^2 + a_3 u^3 + \cdots, \qquad (a_1 \neq 0), \\ V &= v(a_1 + 2a_2 u + 3a_3 u^2 + \cdots)? \end{aligned}$$

The requisite condition is expressed by the identity

* See Scheffers, *Math. Annalen*, 1905. The lines and curves considered are oriented.

$$(7)\quad (\alpha_1 u + \alpha_2 u^2 + \cdots)(a_1 + 2a_2 u + 3a_3 u^2 + \cdots)$$
$$= A_1(a_1 u + a_2 u^2 + \cdots) + A_2(a_1 u + a_2 u^2 + \cdots)^2 \cdots.$$

Equating coefficients, we find first $a_1\alpha_1 = a_1 A_1$. Since $a_1 \neq 0$, it follows that $\alpha_1 = A_1$. Hence α_1 is an absolute invariant. In fact α_1, in the curve (4), is the distance from the origin along the tangent $u = 0$, $v = 0$ to the point of contact with the curve. This verifies the invariance of δ, the tangential distance of two curves.

Equating coefficients of u^n in (7), we find an equation involving $a_1, a_2, \cdots, a_n$, the coefficient of a_n being

$$n\alpha_1 - A_1 = (n-1)\alpha_1 \qquad (n > 1).$$

Hence, if we assume $\alpha_1 \neq 0$, the equation can be solved for a_n. Hence no higher absolute invariants exist.

THEOREM I. *A pair of curves whose tangential distance δ is not equal to zero has no absolute invariant, under the equilong group, except δ.*

To show that two pairs of curves having the same δ are actually equivalent, it is of course necessary to show that the first of the series (6), whose coefficients a_n are calculated as described above, is convergent. This we shall do later.

We consider next the case $\delta = 0$, that is, the case where the two curves form a horn angle.* Let the order of contact of the two curves be $h - 1$, where h may be 2, 3, $\cdots$. In our reduced form one of the curves is the point $v = 0$, so the other must be of the form

$$v = \alpha_h u^h + \alpha_{h+1} u^{h+1} + \cdots \qquad (\alpha_h \neq 0).$$

This takes the place of (4). In the second plane the curve (5) must take the same form, since order of contact is obviously an arithmetic invariant. The transformation (6) remains the same.

The first equation obtained from the identity (7) is now

$$\alpha_h = a_1^{h-1} A_h,$$

which determines a_1.† In the next equation the coefficient of

* See the author's paper cited above.

† If $h-1$ is even, and if α_h and A_h have opposite signs, the a_1 thus found will be imaginary. In this case we may apply a preliminary improper equilong transformation, say $U' = U$, $V' = -V$, which will change the sign of A_h. Hence we can always take a_1 to be real. The other coefficients $a_2, a_3, \cdots$ are found rationally, hence the transformation will be real.

a_2 is $(2-h)\alpha_h$; hence we can solve for a_2 unless $h=2$. In general, the nth equation determines a_n unless $h=n$. Hence for a given order of contact, that is, a given value of h (greater than unity), there is one and only one equation of the set which cannot be solved for one of the coefficients (the one with highest subscript) of the transformation. This particular equation, the hth in the list, together with the previous equations, will enable us to eliminate $a_1, a_2, \cdots, a_{h-1}$, thus giving a relation between the coefficients $\alpha_h, \cdots, \alpha_{2h-1}$ and $A_h, \cdots, A_{2h-1}$ of the two curves. This relation can be separated in the form

$$J_{2h-1}(\alpha_h, \alpha_{h+1}, \cdots, \alpha_{2h-1}) = J_{2h-1}(A_h, A_{h+1}, \cdots, A_{2h-1}),$$

where J is a certain rational function of its arguments. Hence we have an absolute invariant of order $2h-1$.

THEOREM II. *Any horn angle, that is, a pair of curves touching each other, has one and only one equilong invariant. If the order of contact is $h-1$, the order of the absolute invariant J_{2h-1} is $2h-1$.*

If we allow h to take the value unity, the curves will *not* be in contact (we may call this contact of order zero), and the invariant J_1 is merely δ, the tangential distance. It is thus easy to restate Theorem II so as to include Theorem I.

In the case of simple contact, $h=2$, the invariant J_3 is of third order and has the following geometric meaning (not restricted to the canonical form in which one of the curves is reduced to a point)

$$J_3 = \frac{r_1\dfrac{dr_1}{ds_1} - r_2\dfrac{dr_2}{ds_2}}{(r_1-r_2)^2} = \frac{\dfrac{dr_1}{d\theta_1} - \dfrac{dr_2}{d\theta_2}}{(r_1-r_2)^2}.$$

Here r_1 and r_2 denote the radii of curvature of the two curves of the horn angle; ds_1 and ds_2 denote the elements of arc; $d\theta_1$ and $d\theta_2$ denote the changes in the inclination of the tangent. The radii and their rates of change are of course taken at the vertex of the angle.*

* The analogous conformal invariant of a horn angle is (see first citation)

$$I_3 = \frac{\dfrac{d\gamma_1}{ds_1} - \dfrac{d\gamma_2}{ds_2}}{(\gamma_1-\gamma_2)^2},$$

where γ denotes curvature.

2. *Convergence Proofs.*

If two pairs of curves have the same absolute invariant, then it is possible at least formally to find a power series for an equilong transformation converting the one pair into the other. To show that the series thus obtained is always convergent, we restate our equivalence problem (again in its canonical form) in terms of differential equations.

If $v = f(u)$ and $V = F(U)$ are to be equivalent under the transformation $U = \varphi(u)$, $V = v\varphi'(u)$, then

$$\varphi'(u) = \frac{F(\varphi(u))}{f(u)}.$$

Here φ is the unknown function. It will be convenient to replace u by x, and φ by y. Thus our differential equation is

$$\frac{dy}{dx} = \frac{F(y)}{f(x)} = \frac{A_1 y + A_2 y^2 + \cdots}{\alpha_1 x + \alpha_2 x^2 + \cdots}. \tag{8}$$

The curves will be equivalent if, and only if, this differential equation admits an analytic solution of the form

$$y = a_1 x + a_2 x^2 + \cdots \qquad (a_1 \neq 0). \tag{9}$$

The formal conditions are obtained from an identity which is obviously the same as (7) with u replaced by x. Hence we have a single condition on the coefficients in (8), namely, $J(\alpha) = J(A)$. The series (9) then formally exists, in fact there will always be ∞^1 such series, one of the coefficients (namely a_h) being arbitrary.

To show that the series obtained are convergent (that is, that the radius of convergence is greater than zero) we might use directly Cauchy's method of majorants; but this is unnecessary, since we can appeal to the following result due to Briot and Bouquet:* If an equation of the form

* Goursat, Cours d'Analyse, vol. 2 (second edition), pp. 503, 504. It is there shown that if the coefficient a_{01}, or b in Goursat's notation, is not a positive integer, there is one solution; if b is a positive integer, there will be either 0 or ∞^1 solutions. But divergent series never arise.

This is of course not true for differential equations of all forms. For example,

$$\frac{dy}{dx} = \frac{y - x}{x^2}$$

is satisfied formally by $y = x + x^2 + 2!x^3 + 3!x^4 + \cdots$, which is divergent for all values of x other than zero.

$$(10)\qquad \frac{dy}{dx} = \frac{a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots}{x},$$

can be solved formally by a power series

$$y = c_1x + c_2x^2 + \cdots,$$

this series will necessarily be convergent.

We discuss first the case where the tangential distance δ is not zero. Then $\alpha_1 = A_1 \neq 0$, so we may write (8) in the form

$$(11)\qquad \frac{dy}{dx} = \frac{(A_1y + \cdots)(\alpha_1 + \alpha_2x + \cdots)^{-1}}{x} = \frac{y + R(x, y)}{x},$$

where R is a power series beginning with terms of the second degree in x and y. This equation is of the form (10). Hence convergence is assured.

We take next the case $\delta = 0$. If the order of contact is $h - 1$, the differential equation (8) becomes

$$(12)\qquad \frac{dy}{dx} = \frac{A_hy^h + A_{h+1}y^{h+1} + \cdots}{\alpha_hx^h + \alpha_{h+1}x^{h+1} + \cdots}, \quad (\alpha_h \neq 0, A_h \neq 0, h > 1).$$

To reduce this to the Briot and Bouquet form, we make a change in the dependent variable, using the substitution

$$(13)\qquad y = (z + \lambda)x, \quad \text{where} \quad \lambda^{h-1} = \frac{\alpha_h}{A_h}.$$

The transformed equation is found to be

$$(14)\qquad \frac{dz}{dx} = \frac{(h - 1)z + R(x, z)}{x},$$

where the power series R starts with terms of the second degree. This is of the form (10), the coefficient b or a_{01} being the integer $h - 1$. Hence z will be a convergent power series. By (13), the same will then be true of y.

THEOREM III. *The convergence of the series defining the equilong transformation is thus guaranteed in every case.*

In our discussion we have assumed one of the curves of the pair in each plane to be reduced to a point, the origin. We may pass directly from this canonical form to the general form, and state our final result as follows:

THEOREM IV. *If one pair of curves*

$$v = \alpha_1 u + \alpha_2 u^2 + \cdots,$$

$$v = \beta_1 u + \beta_2 u^2 + \cdots,$$

is to be equivalent, under the equilong group, to a second pair of curves

$$V = A_1 U + A_2 U^2 + \cdots,$$

$$V = B_1 U + B_2 U^2 + \cdots,$$

the necessary and sufficient condition is the equality of a single absolute invariant J, that is,

$$J(\alpha, \beta) = J(A, B).$$

If the order of contact of the curves of each pair (this is obviously an arithmetic invariant) is $h - 1$, the invariant J is of order $2h - 1$.

If $h = 1$ (curves not touching), J is the tangential distance δ. If $h = 2$ (simple contact), J is a combination of the radii of curvature and their rates of variation, as given above.

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THE INVERSION OF AN ANALYTIC FUNCTION.

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THE demonstration of the existence of the inverse of an analytic function is made to depend in the Weierstrass theory upon the power series representation of the function and in the Cauchy theory upon the Jacobian of the real and imaginary parts of the function with reference to the real and imaginary parts of the variable. The proof presented in the following pages finds its source in the Goursat conception of an analytic function and is related as to method to the theory of sets of points.

Suppose the function $w = f(z)$ exists and has a finite derivative at each point of a simply connected domain D.

Suppose z_0 is a point of D for which the derivative is different from zero. There is a simple, closed curve C lying entirely in D and containing z_0 as an interior point, such that for points within or on it the values of the function are all different from one another and the values of the derivative are all different from zero. For if $f(z) - f(z') = 0$ can be satisfied by different values z, z' both as near z_0 as desired, then

$$f'(z_0) = \lim_{z, z' \to z_0} \int_\Gamma \frac{f(t)dt}{(t-z)(t-z')}$$

$$= \lim_{z, z' \to z_0} \frac{f(z) - f(z')}{z - z'}$$

$$= 0,$$

the simple closed curve Γ lying entirely in D and containing within it z_0 and the points z, z' as above described; while if the derivative equals zero for values z as near z_0 as desired, then its value at z_0 is also zero. Denote the aggregate of interior points of C by $\{z\}$ and the aggregate of function values resulting therefrom by $\{w\}$. Denote by $\{z\}'$ the derived set of $\{z\}$ and by $\{\alpha\}$ the aggregate of points of $\{z\}'$ not in $\{z\}$. Then $\{z\}' = \{z\} + \{\alpha\}$. But $\{w\}$ is included in $\{w\}'$, for if $\bar{w}$ is an isolated point of $\{w\}$ and if $\bar{z}$ is the corresponding point of $\{z\}$ then $(w - \bar{w})/(z - \bar{z})$ approaches infinity as z approaches $\bar{z}$, which contradicts the assumption of a finite derivative at $\bar{z}$. Denote by $\{\beta\}$ the aggregate of points of $\{w\}'$ not in $\{w\}$. Then $\{w\}' = \{w\} + \{\beta\}$.

On the basis of the relation $w = f(z)$ there is a (1, 1) correspondence between the points of $\{\beta\}$ and $\{\alpha\}$. For suppose α is a point of $\{\alpha\}$ defined by the sequence $z_1, z_2, z_3, \cdots$ made of points of $\{z\}$ and suppose α is the only limit point of the sequence. The corresponding sequence $w_1, w_2, w_3, \cdots$, made up of points of $\{w\}$ must define at least one number b. If the latter sequence has more than one limit point, then by dropping out corresponding elements from each sequence two new sequences are secured defining only the numbers α and b respectively. Proceeding from such sequences it appears that

$$\lim_{n \to \infty} \frac{w_n - f(\alpha)}{z_n - \alpha} = f'(\alpha) \neq 0, \quad \lim_{n \to \infty} z_n - \alpha = 0,$$

from which it follows that $\lim_{n\to\infty} w_n - f(\alpha) = 0$, which associated with the relation $\lim_{n\to\infty} w_n - b = 0$ furnishes as result $b = f(\alpha)$. It is now evident that there was no possibility of the latter sequence defining more than one number. Hence corresponding to any number α of $\{\alpha\}$ is one and only one number β of $\{\beta\}$ so that $\beta = f(\alpha)$. It may be proved in like manner that corresponding to any number β of $\{\beta\}$ is one and only one number α of $\{\alpha\}$ so that $\beta = f(\alpha)$. Finally this same method of proof coupled with the fact that $\{\alpha\}$ includes $\{\alpha\}'$ may be employed to establish that $\{\beta\}$ includes $\{\beta\}'$.

Each point of $\{w\}$ is an inner point of $\{w\}$. For denote the aggregate of all points in the plane not included in $\{w\}+\{\beta\}$ by $\{\theta\}$. Each point of $\{\theta\}$ is an inner point of $\{\theta\}$, since the limit points of $\{w\} + \{\beta\}$ are all included in $\{w\} + \{\beta\}$. It has been proved that no point of $\{w\}$ is a limit point of $\{\beta\}$ and it remains to prove that no point of $\{w\}$ is a limit point of $\{\theta\}$. Suppose that w is any point of $\{w\}$. With w as center draw a circle of radius σ, within which is no point of $\{\beta\}$. Draw a concentric circle of radius $\sigma/2$. This latter circle contains within it no points of $\{\theta\}$. For if so, suppose $\bar{\theta}$ is such a point. There is then a circle with $\bar{\theta}$ as center and of radius $\bar{\sigma} < \sigma/2$ containing within it only points of $\{\theta\}$ and such that any larger concentric circle contains within it points of $\{w\}$ as well. Such circle must have on its boundary a point $\bar{w}$ of $\{w\}$, whose corresponding point is $\bar{z}$ of $\{z\}$. For z sufficiently near to $\bar{z}$, amp $(z - \bar{z})$ can take values differing by any assigned positive number less than 2π, while amp $(w - \bar{w})$ cannot take values differing by any assigned number greater than π. Hence amp $[(w - \bar{w})/(z - \bar{z})]$ is without a limit as z approaches $\bar{z}$, which contradicts the assumption of a derivative at $\bar{z}$, finite and different from zero. An evident extension would serve to show that there are no points of $\{\theta\}$ within the circle of radius σ.

Moreover, any two points of $\{w\}$ can be joined by a simple, regular curve composed entirely of points of $\{w\}$, for the corresponding points of $\{z\}$ can be joined by a simple, regular curve composed entirely of points of $\{z\}$, which on the basis of the relation $w = f(z)$ furnishes the simple, regular curve desired joining the two points of $\{w\}$. The set of points $\{w\}$ is therefore a domain and the set of points $\{z\}$ is connected therewith by a relation of the form $z = \phi(w)$. The

fact that $\phi'(w)$ exists and equals the reciprocal of $f'(z)$ justifies the conclusion that $z = \phi(w)$ is an analytic function of w for the domain $\{w\}$, and this is the inverse of $w = f(z)$ for the domain $\{z\}$.

Suppose next that the derivative is zero at z_0. It is not possible that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^n} = 0 \quad (n = 1, 2, 3, \cdots),$$

unless $f(z)$ is a constant. This is at once evident from the power series representation for $f(z)$, but it is more in line with the subsequent course of the present paper to establish it in the manner now to be described. Where $n - 1$ is any assigned positive integer and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^{n-1}} = 0,$$

define $w_{n-1}(z)$ in D to equal

$$\frac{f(z) - f(z_0)}{(z - z_0)^{n-1}}$$

for z different from z_0 and to equal zero for $z = z_0$. Then $w_{n-1}(z)$ is analytic in D. If C is any simple, closed curve in D containing z_0 as an interior point, then for any other interior point z

$$\frac{w_{n-1}(z)}{z - z_0} = \frac{1}{2\pi i} \int_C \frac{w_{n-1}(t)dt}{(t - z)(t - z_0)},$$

from which it follows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^n} = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t - z_0)^{n+1}}.$$

The proposition is, therefore, to show that it is impossible to have

$$\int_C \frac{f(t)dt}{(t - z_0)^{n+1}} = 0 \quad (n = 1, 2, 3, \cdots),$$

unless $f(z)$ is a constant. If z is any interior point other than the center z_0 of a circle lying entirely within C and if all the integrals above are supposed to be zero, then

$$(z - z_0)f(z_0) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_C \left(\frac{z - z_0}{t - z_0}\right)^n f(t)dt$$

$$= \frac{1}{2\pi i}\int_C \sum_{n=1}^{\infty}\left(\frac{z - z_0}{t - z_0}\right)^n f(t)dt$$

$$= (z - z_0)f(z).$$

That is, $f(z) = f(z_0)$, which contradicts the assumption here made that $f(z)$ is not a constant. Hence for a certain positive integral value of $n - 1$

$$\lim_{z\to z_0}\frac{f(z) - f(z_0)}{(z - z_0)^{n-1}} = 0, \quad \text{but} \quad \lim_{z\to z_0}\frac{f(z) - f(z_0)}{(z - z_0)^n}$$

either does not exist or if so is not zero. The previous argument shows that this latter limit exists and equals

$$\frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t - z_0)^{n+1}},$$

which may be denoted by $\rho e^{i\theta}$. Adopting the definition already given for $w_{n-1}(z)$ it appears that

$$w'_{n-1}(z_0) = \lim_{z\to z_0}\frac{w_{n-1}(z)}{z - z_0} = \rho e^{i\theta}.$$

Substituting ξ^n for $w - f(z_0)$, then for any point z of D other than z_0

$$\left(\frac{\xi}{z - z_0}\right)^n = \frac{w_{n-1}(z)}{z - z_0}.$$

Since

$$\lim_{z\to z_0}\frac{f(z) - f(z_0)}{(z - z_0)^n}$$

exists and is different from zero, z_0 is not a limit point of points z for which $f(z) - f(z_0) = 0$. Since for any point z of D other than z_0

$$f'(z) = (z - z_0)^{n-1}\left[w'_{n-1}(z) + (n - 1)\frac{w_{n-1}(z)}{z - z_0}\right],$$

z_0 is not a limit point of points z for which $f'(z) = 0$, for in that case $w_{n-1}(z_0)$ would equal zero. Hence where d is any fixed positive number less than $\rho \sin(\pi/n)$, a domain R about

but not inclusive of z_0 exists so that for any point z therein ξ^n and dw/dz are both different from zero, and

$$\left|\frac{w_{n-1}(z)}{z-z_0}-\rho e^{i\theta}\right| \leqq d,$$

or what is the same thing,

$$\frac{w_{n-1}(z)}{z-z_0}=\rho e^{i\theta}\left(1+\frac{h}{\rho}e^{i\phi}\right),$$

in which $0 \leqq h \leqq d$. One of the values of $\xi/(z-z_0)$ corresponding is

$$\rho^{1/n}e^{i(\theta+2q\pi)/n}\left(1+\frac{h}{\rho}e^{i\phi}\right)^{1/n}=\rho^{1/n}e^{i(\theta+2q\pi)/n}\left(1+\frac{k}{\rho}e^{i\psi}\right),$$

in which $0 \leqq k \leqq h$. Hence for such a determination of $\xi/(z-z_0)$ it appears that

$$\left|\frac{\xi}{z-z_0}-\rho^{1/n}e^{i(\theta+2q\pi)/n}\right| < \rho^{1/n}\sin\frac{\pi}{n}.$$

Therefore, for z in the domain R there are n determinations of ξ so that the corresponding values of $\xi/(z-z_0)$ lie one within each of n equal circles, each circle lying exterior to the remaining $n-1$ circles and the centers of the circles representing the nth roots of $\rho e^{i\theta}$. Having regard to the values of $\xi/(z-z_0)$ lying within a definite one, but nevertheless any one of these circles as determined by the values of z in R, and noting that $\xi=0$ for $z=z_0$, then for the domain composed of R and the point z_0, ξ is a one-valued function of z. Moreover, the derivative of ξ for $z=z_0$ is the number represented by the center of the particular circle chosen. For any point z of R

$$\frac{d\xi}{dz}=\frac{1}{n}\frac{dw}{dz}\xi^{1-n},$$

which is finite and different from zero. That is, ξ is an analytic function of z for the domain R and the point z_0, and for this latter point the derivative is not zero. Hence by decreasing d, a domain R can be found so that the corresponding values of ξ are all different from one another. Hence corresponding to such a domain R and the point $z=z_0$ is a domain S and the point $\xi=0$, the variables of which are connected by the analytic relations

$$\xi = \phi(z), \quad z = \psi(\xi).$$

It is clear that corresponding to any point w in the vicinity of $f(z_0)$ the function $z = \psi(\xi)$ furnishes n values of z. Also the form of $\psi(\xi)$ would depend on the particular circle chosen, but one form may be transformed into any other by replacing ξ by the product of ξ and the appropriate nth root of unity.

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EMORY McCLINTOCK.

BUT few members of the American Mathematical Society at the present time appreciate the magnitude of the services rendered by its former president, Emory McClintock, who died July 10, 1916.

He was born September 19, 1840, at Carlisle, Pa. His father was the Rev. John McClintock, a learned Methodist Episcopal clergyman, for a time professor of mathematics, Latin, and Greek in Dickinson College, and during the Civil War chaplain of the American Chapel in Paris. He is perhaps best known as the author, with another, of a "Cyclopædia of Biblical, Theological, and Ecclesiastical Literature."

McClintock went to school for the first time at the age of thirteen, and a year later entered the freshman year of Dickinson College. In 1856, when his father left Dickinson College for New York, he entered Yale, and in 1857 he entered Columbia as a member of the class of 1859. His remarkable ability excited the admiration of his teachers, Professors Charles Davies and William Guy Peck. In April, 1859 in order to meet an emergency caused by the illness of a member of the teaching staff, he was graduated and appointed tutor in mathematics. Soon afterwards his father took charge of the American Chapel in Paris, and in 1860 McClintock resigned his position at Columbia to go abroad. In 1861 he studied chemistry at the University of Göttingen. In 1862 he returned to America with the intention of engaging in the Civil War. He was offered an appointment as second lieutenant of topographical engineers in the United States Army, but on his way to Washington suffered a sunstroke which prevented him

from entering the army. From 1863 until 1866 he was consular agent of the United States in Bradford, England. During 1866 and part of 1867 he was connected with a private banking firm in Paris. In 1867 he was appointed actuary of the Asbury Life Insurance Company of New York, and in 1871 he became the actuary of the Northwestern Life Insurance Company of Milwaukee, a position that he retained for eighteen years.

Early in 1889 he was appointed actuary of the Mutual Life Insurance Company of New York to succeed Professor W. H. C. Bartlett. With this company he remained until his death, serving as actuary until 1911, and continuing thereafter as consulting actuary. From 1905 until 1911 he was also vice-president and trustee. In the autumn of 1906 serious ill health compelled him to relinquish a part of his activities and in 1911 he retired altogether from active work for the company.

McClintock applied for membership in the New York Mathematical Society in November, 1889, was elected vice-president the following month and president at the annual meeting in December, 1890. He retired from the presidency at the annual meeting in December, 1894, on which occasion he delivered a very remarkable and interesting address entitled: "The Past and Future of the Society."* At the same meeting the Society adopted the following resolution:

"*Resolved*, That the American Mathematical Society on the occasion of the retirement from the presidency of Dr. Emory McClintock expresses its appreciation of the great services rendered by him while presiding officer of the Society and its recognition of the fact that to his initiative were due the broadening of organization and extension of membership which have made the Society properly representative of the mathematical interests of America."

The writer, who in 1888 had been offered a position in the actuarial department of the Mutual Life Insurance Company, and who was secretary of the Mathematical Society in its early days, made the acquaintance of Dr. McClintock shortly after his arrival in New York in 1889. He frequently visited McClintock at "Kemble Hill," his home near Morristown, New Jersey, and soon came to admire his learning, scientific ability, breadth of view, and far-sighted wisdom. McClintock at all times displayed the keenest and most active

* Published in the BULLETIN, vol. 1, pp. 85–94.

interest in every plan for enlarging the scope of the Society and improving the character of its work. It was chiefly through his encouragement and support that the BULLETIN was founded in 1891. His influence and financial assistance led the Society in 1896 to undertake the publication of the papers read at the International Mathematical Congress in Chicago in 1893. His wise advice, patience in conference, and generous assistance were most helpful to the Society in connection with the establishment of the *Transactions* in 1900.

On one or two occasions the writer heard him express regret that he had not followed an academic career, which would have permitted him to give a larger share of his time to research and enjoy those mutually inspiring relations that so often exist between a teacher and his students. However McClintock never failed to stimulate and inspire everyone of scientific aptitude or taste with whom he came in contact; and as one who was not by profession a teacher of the subject, he stands practically alone among notable American contributors to pure mathematics.

McClintock's scientific publications date from 1868, when he began his epoch-making contributions to the theory of life insurance. For an account of his work in this field the reader is referred to the appreciation recently published in the *Transactions* of the Actuarial Society of America.*

His contributions to the literature of pure mathematics began when adequate facilities were first afforded for the publication of the mathematical researches of Americans by the establishment of the *American Journal of Mathematics* in 1878. Immediately thereafter he submitted several papers for publication. His first paper, entitled "An essay on the calculus of enlargement," was an effort to present the theory of finite differences and the differential calculus from a unified point of view. The paper may be regarded as a precursor of recent attempts to consider difference equations as differential equations of infinite order.

His other more important papers were a series of researches on solvable quintic equations published in the *American Journal of Mathematics* and a paper on the theory of numbers published in the third volume of the *Transactions*. During the years 1879 to 1899 McClintock published fifteen papers in the *American Journal of Mathematics*, being for a number of

* Vol. 17, part II, (Oct., 1916).

years the most frequent American contributor not on the editorial staff of the Journal.

When one considers that McClintock made no use of the powerful labor saving machinery which has revolutionized modern analysis, the results obtained by him in his researches on quintic equations, as well as some of his other achievements, appear to indicate a truly wonderful power of manipulation and clearness of vision.

He received the honorary degree of Ph.D. from the University of Wisconsin in 1884, LL.D. from Columbia in 1885, and LL.D. from Yale in 1899. He was president of the Actuarial Society of America from 1895 to 1897. He was an honorary fellow of the American Academy of Arts and Sciences, a member of the London Mathematical Society, and a fellow of the Institute of Actuaries.

A list of McClintock's mathematical publications is given below:

1. "An essay on the calculus of enlargement." *American Journal of Mathematics*, volume 2 (1879), pages 101–161.
2. "A new general method of interpolation." *American Journal of Mathematics*, volume 2 (1879), pages 307–314.
3. "On a theorem for expanding functions of functions." *American Journal of Mathematics*, volume 2 (1879), pages 348–353.
4. "Note on a theorem for expanding functions of functions." *American Journal of Mathematics*, volume 3 (1880), page 173.
5. "On certain expansion theorems." *American Journal of Mathematics*, volume 4 (1881), pages 16–24.
6. "On the remainder of Laplace's series." *American Journal of Mathematics*, volume 4 (1881), page 96.
7. "On the resolution of equations of the fifth degree." *American Journal of Mathematics*, volume 6 (1884), pages 301–315.
8. "Analysis of quintic equations." *American Journal of Mathematics*, volume 8 (1885), pages 45–84.
9. "On the algebraic proof of a certain series." Presented to the Society March 6, 1891. *American Journal of Mathematics*, volume 14 (1891), pages 67–71.
10. "On independent definitions of the functions log (x) and e^x." Presented to the Society March 6, 1891. *American Journal of Mathematics*, volume 14 (1891), pages 72–86.
11. "On lists of covariants." *Bulletin of the New York Mathematical Society*, volume 1 (1892), pages 85–91.
12. "On the computation of covariants by transvection." Presented to the Society January 2, 1892. *American Journal of Mathematics*, volume 14 (1892), pages 222–229.
13. "On the non-euclidian geometry." *Bulletin of the New York Mathematical Society*, volume 2 (1892), pages 21–33.
14. "On the early history of the non-euclidian geometry." *Bulletin of the New York Mathematical Society*, volume 2 (1893), pages 144–147.
15. "Theorems in the calculus of enlargement." Presented to the Society August 14, 1894. *American Journal of Mathematics*, volume 17 (1895), pages 69–80.

16. "A method for calculating simultaneously all the roots of an equation." Presented to the Society August 14 and October 27, 1894. *American Journal of Mathematics*, volume 17 (1895), pages 89–110.
17. "The past and future of the Society." Presidential Address, delivered before the Society December 28, 1894. *Bulletin of the American Mathematical Society*, volume 1 (1895), pages 85–94.
18. "On the most perfect forms of magic squares, with methods for their production." Presented to the Society April 25, 1896. *American Journal of Mathematics*, volume 19 (1897), pages 99–120.
19. "On a solution of the biquadratic which combines the methods of Descartes and Euler." Presented to the Society May 29, 1897. *Bulletin of the American Mathematical Society*, volume 3 (1897), pages 389–390.
20. "Further researches in the theory of quintic equations." Presented to the Society August 17, 1897. *American Journal of Mathematics*, volume 20 (1898), pages 157–192.
21. "A simplified solution of the cubic." Presented to the Society December 28, 1900. *Annals of Mathematics*, series 2, volume 2 (1901), pages 151–152.
22. "On the nature and use of the functions employed in the recognition of quadratic residues." Presented to the Society December 27, 1901. *Transactions of the American Mathematical Society*, volume 3 (1902), pages 92–109.
23. "The logarithm as a direct function." Presented to the Society February 28, 1903. *Bulletin of the American Mathematical Society*, volume 9 (1903), pages 467–469.

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ON FOCI OF CONICS.

BY DR. J. H. WEAVER.

THE study of the properties of conic sections, and of certain related points and lines has afforded ample scope for the faculties of all mathematicians from the Greeks on down through the ages. And not the least important among the points connected with the sections are the foci. It is the object of the present paper to give (I) a short historical sketch of the development of the properties of conics connected with the foci; (II) some of the theorems from Pappus which have a bearing on foci and tangents.

I. *Historical Sketch.*

What name should be connected with the discovery of the foci is still a matter of conjecture. Zeuthen* seems to think

* Geschichte der Mathematik im Alterthum und Mittelalter, Kopenhagen, 1896, p. 211.

that the focus for the parabola was known to Euclid. However, we have no mention of such points or of any of their properties until we arrive at the time of Apollonius (about the middle of the second century B.C.). He proves for us in his Conics, Book III, Props. 45–52,* the following properties of foci for central conics.

I. If Ar and $A'r'$, the tangents at the extremities of the axis of a central conic, meet the tangent at P in r and r' respectively, then

(1) rr' subtends a right angle at each focus S and S';

(2) $\angle rr'S = \angle A'r'S'$ and $\angle r'rS' = \angle ArS$.

II. If O is the intersection of rS' and $r'S$, then OP is perpendicular to the tangent at P.

III. The focal radii of P make equal angles with the tangent at that point.

IV. If from either focus as S, SY be drawn perpendicular to the tangent at any point P, the angle AYA' will be a right angle, or the locus of Y is a circle on AA' as diameter.

V. If C is the center of the conic, a line drawn through C parallel to either of the focal radii of P to meet the tangent will be equal in length to CA.

VI. In an ellipse the sum, and in a hyperbola the difference, of the focal distances from any point on the conic is equal to the axis AA'.

Apollonius does not use or mention in any way the focus for the parabola.

The next mention that we have of foci is given us by Pappus (about the end of the third century A.D.). He gives us the first recorded use and proofs of the focus-directrix definition of conics.† In addition to these Pappus has several other lemmas that have a bearing on foci. Their statement and proof will be given in section II.

But although Apollonius and Pappus have recorded for us the most notable of the properties of the foci, neither of them attached any name to the points in question. That honor was left for Johann Kepler (1571–1630). In his work Ad Vitellionem Paralipomena quibus, Astronomæ Pars Optica Traditur, Francofurti, 1604,‡ he gives a short account of the conic sections, a part of which runs as follows:

* See Conics of Apollonius, ed. Heath, pp. 113–118.

† For a discussion of these theorems see my article, Pappus, "Introductory Paper," BULLETIN, vol. 23, No. 3, p. 134.

‡ See Kepler, Opera Omnia, ed. Frisch, Frankofurti, 1859, vol. II, p. 185.

"There are among these curves certain points of especial consideration, which have a certain definition but no name, unless they usurp for name the definition or some property. For if from these points lines are drawn to the points of contact of tangents to the section, these lines make equal angles with the tangents. . . . We, because of the properties of light and the eye, from the viewpoint of mechanics shall call these points foci. We might have called them centers, because they are on the axis of the section, if authors, in the hyperbola and ellipse, were not accustomed to calling another point the center. In the circle there is one focus, the center. In the ellipse there are two foci equally distant from the center, and more removed in the more acute. In the parabola, one focus is within the section and the other may be considered either within or without the section and removed to an infinite distance from the first focus, so that if a line drawn from this '*cæcus*' focus to a point of the section will be parallel to the axis. In the hyperbola, the external focus becomes nearer the internal focus as the hyperbola becomes more obtuse."

In the above rather free translation of Kepler's remarks two things are to be noted as distinct contributions to geometry: (1) The discovery that the parabola may be considered as having two foci; (2) the formulation of the idea that parallel lines are concurrent at a point at infinity.

Following hard upon the footsteps of Kepler, Girard Desargues (1593–1662) extended the notion of the new doctrine of infinity, used the general method of projection and gave methods for determining the foci of a conic both in a plane and when the conic is in a cone. In the plane case he used the circle defined by Apollonius in his conics.* The method for determining the foci in a cone has been summarized by Chasles as follows:

Given any conic O and a cone through it. Let O' be any section of the cone. Through the vertex V pass a plane parallel to that of O' meeting the plane of O in the line ab. Take any point t on ab and let the chord of contacts of the tangents from t to O meet ab in t'. Also let rr' be any segment of ab which subtends a right angle at V. The two sets of points tt' and rr' constitute two involutions having one segment cc' in common. The polars X, X' of c and c' correspond to the axes of O'. . . . The tangents to O from the points r and the lines from r' to their several points of contact determine on

* See I (1) above.

X an involution, whose double points correspond to the foci of O', since every tangent and its normal are harmonic conjugates to the focal distances of the point of contact.*

Another interesting determination for foci was given by Maclaurin (1698–1746).† It is as follows: Let there be a conic section and let its major axis be TT', and let its auxiliary circle be drawn. Let a tangent be drawn to the conic cutting the auxiliary circle in the points A and B. Let AC be a diameter of the auxiliary circle. Let BC be drawn and let it cut TT' in F. Then F is a focus of the conic.

Poncelet (1788–1867), however, greatly extended the theory of the foci when he worked out his theory of ideal chords, and showed that all circles in a plane pass through the same two points φ and φ' on the line at infinity, and indicated the bearing of these points on the foci of conics.‡ He also shows that an analytic calculation of the foci from their definition gives not two foci for each central conic, but four, two real ones on the major or transverse axis and two imaginary ones on the minor or conjugate axis.§ Plücker (1801–68) extended the ideas of Poncelet on foci to plane curves of all orders, regarding as a focus of any curve the point of intersection of any two tangents drawn to it from φ and φ', one from each.||

A great many properties of foci and methods for determining them have been discovered in modern times. An extensive list of references bearing on this subject may be found in Taylor's Conics Ancient and Modern and in the Encyklopädie der Mathematischen Wissenschaften, Band III_2, Heft I, pages 52–56. Only one of the theorems will be noted here. It is as follows: If any cone is cut by a plane and spheres are inscribed in the cone tangent to the plane, these will touch the plane in the foci of the conic section. One would expect to find such a theorem among those enunciated by the Greeks, but it was not stated in definite form until discovered by Dandelin in 1822.¶

* For a further discussion of the work of Desargues see C. Taylor, "Conics, Ancient and Modern," Cambridge, 1881, pp. lxi and 261, where references to the original sources may be found.

† Geometrica Organica, sive descriptio linearum curvarum universalis, London, 1720, Sec. III, p. 102. See also Poncelet, Traité des Propriétés projectives des Figures, Paris, 1865, Tome I, p. 249.

‡ See Traité, sections 89–98, 453 of tome I.

§ *Annales de Mathématiques*, vol. 8 (1818), p. 222.

|| *Crelle's Journal*, vol. 10, pp. 84–91.

¶ *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-lettres de Bruxelles*, tome II, pp. 171–202.

II. *Some Theorems of Pappus which have a Bearing on Foci.*

In addition to the above mentioned theorems of Pappus relative to the subject the following ones have some interesting properties which have a bearing on the foci. They are lemmas on the books of Determinate Section of Apollonius.*

THEOREM 1: "Let there be a semicircle AEB on the diameter AB (Fig. 1), and in this the perpendiculars CE and DZ and

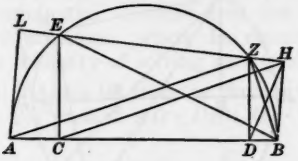

FIG. 1.

let the line EZH be drawn and to this the perpendicular BH, then three things follow: $CB \cdot BD = BH^2$, $AC \cdot DB = ZH^2$ and $AD \cdot BC = EH^2$.

"Let HC, HD, AZ and ZB be drawn. Since $\angle AZB$ is right and DZ is perpendicular to AB, then $\angle DZB = \angle BAZ$. But since the angles BDZ and ZHB are right, the points D, Z, H and B are concyclic, and $\angle DZB = \angle DHB$; then, EB being drawn, $\angle BAZ = \angle BEZ$, being angles in the same segment BZ; and in the segment BH $\angle BEZ = \angle BCH$, therefore $\angle DHB = \angle BCH$. Therefore the triangles CBH and HBD are similar and

$$CB : BH = BH : BD \quad \text{or} \quad BC \cdot BD = BH^2.$$

But also $AB \cdot BD = BZ^2$, whence if we subtract $BC \cdot BD = BH^2$, there remains $AC \cdot BD = ZH^2$. Again since $AB \cdot BC = BE^2$, if we subtract $CB \cdot BD = BH^2$, there remains $AD \cdot CB = EH^2$."

In this proof the equation $BC \cdot BD = HB^2$ signifies that HB is tangent to a circle which passes through C and D and has its center at the midpoint of EZ. If we extend EZ to L, then from the equation $AC \cdot BD = ZH^2$ we have $LE = ZH$, and the circle through D, C and H passes through L. We may then consider AB as the tangent to an ellipse whose major axis is HL, whose foci are E and Z and whose auxiliary circle is the circle through C, D, L and H, while AB is the

* See Pappus, ed. Hultsch, Book VII, Prop. 59, 61, 62 and 64.

portion of a tangent cut off between the two tangents at the vertices of the ellipse, and the circle on AB as diameter is then the circle mentioned by Apollonius as passing through the foci of an ellipse.

THEOREM 2: "Three straight lines AB, BC and CD are given (Fig. 2); then if $AB \cdot BD : AC \cdot CD = BE^2 : EC^2$,

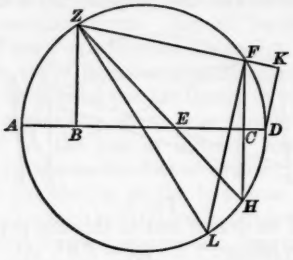

FIG. 2.

$AE \cdot ED : BE \cdot EC$ is a unique and minimum ratio and equals $AD^2 : (\sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)})^2$.

"Let there be described on the diameter AD a circle, and from the diameter let the perpendiculars BZ and CH be drawn to the circumference. Since by hypothesis $AB \cdot BD : AC \cdot CD = BE^2 : EC^2$, and since $AB \cdot BD = BZ^2$ and $AC \cdot CD = CH^2$, then $BZ^2 : CH^2 = BE^2 : EC^2$, and so $BZ : CH = BE : EC$. Therefore the triangles ZBE and HCE are similar and $\angle ZEB = \angle HEC$, and therefore the line through Z, E and H is straight. Let ZEH be drawn and let HC be produced to F, a point of the circumference, and let ZF be drawn and produced to K, whence a perpendicular KD is drawn to ZK; then by the above theorem 1, $AC \cdot BD = ZK^2$, and $AB \cdot CD = FK^2$, therefore ZF (that is $ZK - EK$) $= \sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)}$. Now let the diameter ZL be drawn, and FL also; then $\angle ZFL = \angle ECH$, and in the segment ZF, $\angle ZLF = \angle ZHF$. Therefore the triangles ZFL and ECH are similar. Therefore $LZ : ZF = HE : EC$; that is, because LZ and AD are diameters, $AD : ZF = HE : EC$, and so $AD^2 : ZF^2 = HE^2 : EC^2$; that is, on account of the similar triangles HCE and ZBE which make $HE : EC = ZE : EB$, $AD^2 : ZF^2 = HE \cdot ZE : EC \cdot EB = AE \cdot ED : BE \cdot EC$. And

the ratio $AE \cdot ED : BE \cdot EC$ is unique and minimum, and we have shown above that $ZF = \sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)}$; therefore $AE \cdot ED : BE \cdot EC = AD^2 : (\sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)})^2$ is a unique and minumum ratio."

In this theorem as in the above, AD may be considered as a tangent to an ellipse which has Z and F for foci and K for a vertex. Moreover E is the point of contact of AD with the ellipse. This is evident if we notice that $\angle ZEB = \angle HEC = \angle CEF$. Pappus, however, does not prove that the ratio is actually a minimum. Fermat does this in his theory of maximum and minimum, where he remarks that Pappus recognized the difficulty of solving such problems.*

THEOREM 3: "Again let there be three given lines AB, BC and CD (Fig. 3); then if $AD \cdot DB : AC \cdot CB = DE^2 : EC^2$,

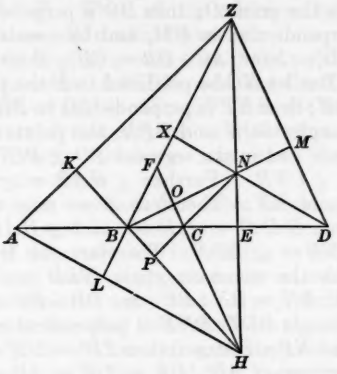

FIG. 3.

$AE \cdot EB : CE \cdot ED$ is a unique and minimum ratio and is equal to $(\sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)})^2 : DC^2$.

"Let the perpendicular EZ be drawn from E to AD and produced so that $AD \cdot DB = ZD^2$ and let HC be drawn parallel to ZD. Since by hypothesis $AD \cdot DB : AC \cdot CB = DE^2 : EC^2$ and in the similar triangles ZED and HEC, $DE : EC = DZ : CH$, then $AD \cdot DB : AC \cdot CB = DZ^2 : CH^2$; and because of the construction $AD \cdot DB = DZ^2$, $AC \cdot CB$

* See Oeuvres de Fermat, Paris, 1896, vol. 3, p. 134.

$= CH^2$. Let AZ, ZB, AH, HB be drawn and let CH be produced and cut the line BZ in F. Since $AD \cdot DB = DZ^2$, $\angle BZD = \angle ZAB$. But because $AC \cdot CB = CH^2$, or $AC : CH = CH : CB$, the angles BHC and BAH are equal. But since DZ and HF are parallel, $\angle BZD = \angle BFH$. Let HB and ZB be produced, and let HB cut AZ in K and ZB cut AH in L; then $\angle BFH + \angle BHF = \angle KBZ$. Moreover $\angle BFH = \angle BZD = \angle ZAB$ and $\angle BHF$ or $BHC = \angle BAH$; therefore $\angle ZAB + \angle BAH = \angle KBZ = \angle KAL$. But $\angle KBL + \angle KAL =$ two right angles, and so the points A, L, B, and K are on a circle. Therefore the angles at L and K are right.* Now let BM be drawn perpendicular to ZD, and let it cut the line EZ in N, and let DN be drawn and produced to X, a point of BZ. Then DX is perpendicular to ZL (see footnote below) and is perpendicular to HL. Let HF cut BN in the point O; then HO is perpendicular to BM (for ZD is perpendicular to BM, and by construction HF is parallel to ZD). Now $AC \cdot CB = CH^2$; therefore $\angle BHC = \angle HAC$. But let NC be produced to P the point of intersection with BH; then NP is perpendicular to BH. Therefore by similar triangles BOH and BPN, the points O, P, H and N are on a circle and in the segment PO $\angle PHO = \angle PNO$, or $\angle BHC = \angle CNB$. Further $\angle HAB = \angle BDN$ in the parallels HA and DX. Therefore if we note that $\angle BHC = \angle CNB$ and $BHC = \angle HAC$, and $\angle HAB = \angle BDN$, whence $\angle CNB = \angle BDN$. Therefore the triangles BNC and BDN with the common angle NBD are similar and in these $DB : BN = BN : BC$ or $DB \cdot BC = BN^2$. But since in the triangle BDZ, DNX is perpendicular, and to this the lines ZN and NB are drawn, then $ZD^2 - DB^2 = ZN^2 - NB^2$. But from construction $AD \cdot DB = ZD^2 = AB \cdot BD + BD^2$; then $ZD^2 - DB^2 = AB \cdot BD$, and so $ZN^2 - NB^2 = AB \cdot BD$. But we have proved that $NB^2 = DB \cdot BC$ and so $ZN^2 = AB \cdot BD + DB \cdot BC = AC \cdot BD$. Therefore $ZN = \sqrt{(AC \cdot BD)}$.

"Again as above $HN^2 - NB^2 = HC^2 - CB^2$. But because as above $AC \cdot CB = HC^2 = AB \cdot BC + BC^2$, $HC^2 - BC^2 = AB \cdot BC$, and so $HN^2 - NB^2 = AB \cdot BC$. But we have shown that $NB^2 = DB \cdot BC$ and so $HN^2 = AB \cdot BC + DB \cdot BC = AD \cdot BC$ or $HN = \sqrt{(AD \cdot BC)}$. Therefore $ZN + NH = \sqrt{(AD \cdot BC)} + \sqrt{(AC \cdot BD)}$.

* Pappus proves this in a preceding lemma. The proof offers no difficulty. See Pappus, Book VII, Prop. 60.

"Again since the angle ZKH is right and AE perpendicular to ZH, by similar triangles AZE and HBE (each being similar to the triangle ABK), then $AE : EZ = HE : EB$, or $AE \cdot EB = ZE \cdot EH$. Therefore $AE \cdot EB : CE \cdot ED = ZE \cdot EH : CE \cdot ED$. But because ZD and HC are parallel, $ZE : ED = HE : EC$ and from this $ZH : CD = ZE : ED$; therefore $ZE \cdot EH : DE \cdot EC = ZE^2 : ED^2 = ZH^2 : CD^2$. Therefore $AE \cdot EB : CE \cdot ED = ZH^2 : CD^2$, and the ratio $AE \cdot EB : CE \cdot ED$ is singular and minimum, and so, since $ZH = \sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)}$, the ratio $(\sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)})^2 : CD^2$ is unique and minimum."

In the above theorem let us assume the line AD to be tangent to a parabola at the point E, and let O be the vertex of the parabola, ON the axis and Z the point in which the normal EN cuts the parabola, and let AZ be the tangent to the parabola at the point Z. Then all the relations in the proof of the above theorem will readily follow.

In theorem 64 Pappus proves that if the points A, B, C, D and E have the order $ABCDE$ on a line then $AD^2 : (\sqrt{(AC \cdot BD)} + \sqrt{(AB \cdot CD)})^2 = AE \cdot ED : BE \cdot EC$ is a unique and maximum ratio and this may be interpreted in terms of the hyperbola in exactly the same way that theorem 2 was interpreted for the ellipse.

Whether Pappus or Apollonius were thinking in terms of conic sections or not in these three problems seems rather doubtful, but if they were it is interesting to note that in the case of the ellipse and the hyperbola use is made of the foci while in the parabola no use is made of any such point. It is also interesting to note that these three are the only maximum and minimum problems (according to Pappus) in the second book of Determinate Section, and that these three cases may be made to depend upon the three conic sections.

WEST CHESTER, PA.

NEW LIGHT ON OUR NUMERALS.

BY MR. JEKUTHIAL GINSBURG.

Introductory Note.—It is interesting to see how much new light is constantly being thrown upon chapters in the history of mathematics which have always been more or less obscure. We know, for example, with reasonable certainty the original habitat of our numerals; we know approximately the century in which they were perfected; we have rather positive information as to the century in which they first appeared in European manuscripts; and we are well advised, through the work of Mr. G. F. Hill, as to their variations in form for the last thousand years. It is true that we do not know when or where the zero of our system was first conceived, although we now have some valuable information as to the one that was used by the Mayas, nor do we know the origin of six of the primitive forms of the digits. Furthermore we do not know with any certainty the date of the first appearance of our numerals on the Mediterranean littoral, but we are not without hope that all this information will sometime be forthcoming, at least to some degree.

Our hope that such further knowledge is not beyond our reach is strengthened by a discovery recently made by M. F. Nau, no report of which seems as yet to have appeared in English. Because of the importance of this discovery, I have asked Mr. Ginsburg to make it known to the readers of the BULLETIN and to supplement the simple statement of the discovery by searching out such information as is available concerning the interesting scholar and teacher, Severus Sebokht, in whose writings the first positive trace of the numerals, outside of India, is found. This he has done, and his article is to my mind particularly valuable because of these features: (1) It shows us that these numerals reached the Arab lands a century earlier than was formerly supposed; (2) it shows that the zero was probably not in the system as then mentioned, showing at least that its value was not generally comprehended in the seventh century and possibly confirming the impression that the symbol had not yet been

invented; (3) it reveals something of the life of a man hitherto unmentioned in the histories of mathematics.

It is to be hoped that this valuable information may prove of such interest to readers that Mr. Ginsburg may be encouraged to tell American scholars, in the near future, something of Sebokht's notable contributions to the study of the astrolabe.

DAVID EUGENE SMITH.

THAT our common numerals are of Hindu origin seems to be a well-established fact,* and that Europe received them from the Arabs seems equally certain, but how and when these numerals reached the Arabs is a question that has never been satisfactorily answered. It is the object of the present article to call the attention of students of the history of mathematics to newly discovered evidence† showing that the Hindu numerals were known to and justly appreciated by the Syrian writer Severus Sebokht who lived in the second half of the seventh century; that is, about a hundred years before the date of the first definite trace that we have hitherto had of the introduction of the system into Bagdad.‡ It will also be shown, on the basis of such information as is available respecting his life and works, that Sebokht was in the most favorable position for getting information of this kind, and that he furthermore had in his possession the most powerful means for the propagating of such knowledge.

Severus Sebokht of Nisibis, bearing the title of bishop, lived in the convent of Kenneshre on the Euphrates§ in the time of the patriarch Athanasius Gammala (who died in 631) and his successor John.‖ He distinguished himself in the studies of philosophy, mathematics, and theology, and in his time the convent of Kenneshre became the chief seat of Greek learning in western Syria. Of his astronomical and geograph-

* Smith and Karpinski, The Hindu-Arabic Numerals, Boston, 1911.

† By the French orientalist M. F. Nau in the *Journal Asiatique*, series 10, vol. 16 (1910).

‡ Smith and Karpinski, The Hindu-Arabic Numerals, p. 92.

§ W. Wright, Short History of Syriac Literature, London, 1894, pp. 137–139.

‖ Sebokht took part, together with the Jacobite patriarch Theodorus, in a public dispute against the Maronites in the year 659. We have also a letter written by him in the year 665. From these details we may conclude that he flourished in the beginning of the second half of the seventh century. (M. F. Nau, in the *Journal Asiatique*, series 9, vol. 13, p. 60.)

ical works there are a few fragments in a manuscript now in the British Museum.* These fragments consider such questions as whether the heaven surrounds the earth in the form of a wheel or of a sphere; the habitable and uninhabitable portions of the earth; the measurement of the heaven, the earth, and the space between them; and the motion of the sun and the moon. His treatise on the plane astrolabe was published with a French translation by M. F. Nau in the *Journal Asiatique*, series 9, volume 13. Sebokht also wrote a short treatise on eclipses, in which he ridicules the then accepted belief in a celestial dragon as the cause of all such phenomena.†

But the most interesting of Sebokht's writings for the student of history is undoubtedly a fragment of a manuscript‡ published by M. F. Nau, in the *Journal Asiatique* (series 10, volume 16, page 225) in which he directly refers to the Hindu numerals. He seems to have been hurt by the arrogance of certain Greek scholars who looked down on the Syrians, and in defending the latter he claims for them the invention of astronomy. He asserts the fact that the Greeks were merely the pupils of the Chaldeans of Babylon, and he claims that these same Chaldeans were the very Syrians whom his opponents condemn. He closes his argument by saying that science is universal and is accessible to any nation or to any individual who takes the pains to search for it. It is not therefore a monopoly of the Greeks, but is international.

It is in this connection that he mentions the Hindus by way of illustration, using the following words: "I will omit all discussion of the science of the Hindus, a people not the same as the Syrians; their subtle discoveries in this science of astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; their valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. If those who believe, because they speak Greek, that they have reached the limits of science should know these things they would be convinced that there are also others who know something." This fragment clearly shows that not only did Sebokht know something of the numerals, but

* Add. 14, 538, pp. 153–155.

† See Notes d'Astronome Syrienne, *Journal Asiatique*, series 10, vol. 16 (1910).

‡ Ms., Syriac, Paris No. 346.

that he understood their full significance, and may even have known the zero as Rabbi ben Esra did, in spite of the fact that he, too, speaks of nine numerals. There are two questions that may immediately arise: (1) How could Sebokht have obtained any information about the Hindu numerals? and (2) What are the chances that Sebokht was instrumental in introducing the numerals to the Arabian scholars?

The first of these questions may be answered very easily. Nisibis, the place where Severus lived, was the chief city* of Mygdonia, a small district in the northeast part of Mesopotamia. It was situated in a rich and fruitful country, was long the center of a very extensive trade, and was the great northern emporium for the merchandise of the east and the west. Since the exchange of goods is always accompanied by the exchange of ideas, it is only reasonable to surmise that the different systems of numeration were known in Nisibis, where they could hardly escape the attention of a man like Sebokht, who would surely have been looking for just such information.

The second question is more difficult to answer. It may be said, however, that the weight of the evidence is in favor of Sebokht's work being at least one of the agencies by means of which the knowledge of the numerals was transmitted to the Arabs. He was the head of his convent and occupied a commanding position in the literature of his country. He had many pupils, one of whom, Athanasius of Balad,† was the patriarch of the Jacobites, while such others as Jacob of Edessa‡ and probably George, Bishop of the Arab Tribes,§ were well known as translators and polygraphers. We may be certain that the knowledge of the numerals possessed on the banks of the Euphrates by Severus was transmitted by him to his numerous pupils and through them to other scholars all over Syria. Since we know that Syrian scholars were employed by the caliphs as translators and educators,‖ it would be only natural that these Syrians should impart to the Arabs, among other facts relating to the sciences, the knowledge of the Hindu numerals.

COLUMBIA UNIVERSITY.

* See Smith's Dictionary of Greek and Roman Geography.

† W. Wright, Short History of Syriac Literature, pp. 154–155.

‡ Ibid., pp. 141–154.

§ Ibid., pp. 156–159; M. F. Nau in the *Journal Asiatique*, series 10, vol. 16.

‖ Ernest Renan, Islamisme et la science, p. 9.

SHORTER NOTICES.

Théorie générale des Nombres. By E. DUMONT, Capitaine du Génie Belge. Paris, Gauthier-Villars, 1916. (No. 35 of the series of Scientia.) 93 pp. 2 francs.

IN spite of the distractions on the Yser in 1914–1915, Captain Dumont has been able to produce a very clear account of the fundamental notions of number which he had previously published in extended form in his book Arithmétique générale (1911). This small volume is one of the physico-mathematical series in the Scientia publications of Gauthier-Villars, which undertake to present in succinct form the ideas of science in process of development. Several others of the series have dealt with recent ideas in geometry, analysis, and algebra.

This volume is divided into two parts: Des Nombres absolus, and Des Nombres relatifs. In the first part is given a development of numbers as the ratios of segments of a straight line. The notion of segment of a line is taken as an undefined element. An absolute number is then any law of formation of a segment from a given segment. The product of a segment by a number (absolute number) is another segment. Captain Dumont in fact sets up a set of logical relatives of which the domain is the range of segments of a straight line, the converse domain is the same range, and the relative is considered only with reference to certain processes of addition, subtraction and multiplication of the segments. These processes are supposed to be known. This first part does not differ essentially, save in formal statements, from other derivations of the system of real numbers.

In the second part the fundamental entity is no longer a segment but a vector, by which is meant a directed segment. The notion of direction is assumed. Definitions are set up for free vectors, glissants, symmetric or opposite vectors, and nul vectors. The definition of relative number then follows as any law of determination of one vector from another; by which is really meant again a logical relative whose two domains are the same, namely the range of all vectors. We arrive thus at qualified numbers, which change a glissant vector into another on the same line, or support; versors, which turn glissant vectors around an axis perpendicular to

the line of the vectors; glisseurs, which move a glissant vector parallel to itself along an axis perpendicular to the line of the vector. Combinations of these give quaternions, in which are included complex numbers, vector-quaternions, and surcomplex numbers.

In this part the author insists on the distinction between the vector that is commonly associated with a complex number in the Argand representation, and the number itself. This distinction is, in the opinion of the reviewer also, vital in many places, and unless noticed leads to trouble. For instance, the Steinmetz treatment of alternating currents is handled by Cramp and Smith from this point of view to the improvement of the whole treatment. Hamilton was the first to define a quaternion as the ratio of two vectors, having arrived at this definition by a study of certain groups of linear substitutions, of which one was a form of the quaternion group as an abstract group. The later expositions given by Hamilton used this geometric method of approach. One might go farther and produce a system of numbers corresponding to the ratios of the members of any set of geometric elements, such as bipoints, biplanes, as Cailler and others have done. Some of these geometric elements have even received names, as flèche, bouclier, drapeau, feuillet, moulinet. Captain Dumont, however, develops the notion only as far as biquaternions.

He insists rightly that the method of development is synthetic, and demands no postulates except those of euclidean space, while other methods of the arithmetico-logical type demand large lists of postulates. Further the development is not referred to any coordinate system, and to no fixed base of units. It becomes thus autonomous. Whether he is correct in thinking it could be taught easily from this intuitive point of view is a question which time and experience will settle. The intuitive method seems at least as good as any way to identify the fundamental elements that one starts with, and in a genuinely logical sense it is just as rigorous. When anyone thinks that by talking of the set of elements $A, B, C, \cdots$ he is any better able to identify an element than he is when he talks of the segments of the line, $OA, OB, OC, \cdots$ he is deceiving himself into a fancied rather than a real rigor of treatment. The really essential thing, it would seem, is whether or not some unstated quality of the element is

one that the deductions depend upon. To illustrate, one may prove the theorems of geometry relating to the circle by using wire circles just as well as by using a highly artificial definition of an abstract circle, provided that he does not depend upon the copper or silver, or their qualities, nor upon the thickness, or cross-section of the wire. In most work in mathematics we are engaged in finding the deductions that can be drawn from certain features only of a concrete existence and which are not in the least affected by other features of the object. In other words the most highly concrete object is just as good as the most highly abstract object for deductions that are limited to certain characters possessed in common by the two. From this point of view the definition from geometric objects is justifiable. On the other hand the recognition of the non-geometric character of all the numbers deduced is one that if more common would prevent much wasted ink and time. Such a notion as that of the product of two geometric vectors ought to disappear from the field, for instance, save as a phraseology perhaps. The fruitless discussions over the identification of the right quaternion and the vector (geometric) would no longer be heard of, and all vector systems would be recognized as algebras of hypernumbers.

JAMES BYRNIE SHAW.

Napier Tercentenary Memorial Volume. Edited by CARGILL GILSTON KNOTT. London, Longmans, Green and Company, 1915. xi+441 pp. Price, £1 1s.

THIS sumptuous volume compiled by Dr. Knott is made up of the addresses and essays communicated to the International Congress held in Edinburgh in 1914 to commemorate the tercentenary of the publication of Napier's epoch-making Mirifici Logarithmorum Canonis Descriptio.

The papers are both historical and mathematical, the former dealing with the life and works of Napier and of his contemporaries, immediate predecessors, or followers, and the latter part of the work treating of the modern progress in calculation, in the preparation of tables, and the like. There is also an account of the Edinburgh meeting, with the addresses of a formal congratulatory nature, a list of members, and two indexes, one of subjects and the other of names.

The historical papers are as follows: "The invention of logarithms," by Lord Moulton, a careful study of the working

of Napier's mind in the development of his theory, made with all the acumen of one of the best legal minds of the present time; "Merchiston Castle," by George Smith, then the headmaster of Merchiston Castle School, the best description of the building that has been published, illustrated by a colored print; "Logarithms and computation," by Dr. J. W. L. Glaisher, partly historical but chiefly given to the post-napierian development; "The law of exponents in the works of the sixteenth century," by the present reviewer; "Algebra in Napier's day and alleged prior inventions of logarithms," by Professor Florian Cajori, a careful and scholarly investigation of the claims of others with a judicial decision against such claims; "Napier's logarithms and the change to Briggs's logarithms," by Professor G. A. Gibson, a critical study of the important question of the relation between Napier and Briggs in the introduction of the base 10; "The introduction of logarithms into Turkey," by Lieut. Salih Mourad, wherein it is shown that logarithms found their way into Turkey in 1714, and that they were explained in the Turkish language in 1765, through the efforts of Ismail Effendi; "A short account of the treatise 'De arte logistica,'" by Professor J. E. A. Steggall, with a facsimile of one of the pages; "The first Naperian logarithm calculated before Napier," by Professor G. Vacca, in which it is shown that Pacioli (1494), in a problem on compound interest, gave a rough calculation of log 2 in solving what is practically $(1 + r/100) = 2$; but that Pacioli had any real conception of logarithms is not asserted; "The theory of Naperian logarithms explained by Pietro Mengoli (1659)," by Professor G. Vacca, in which it is shown that Mengoli, a pupil of Cavalieri, set forth his explanation in the Geometria Speciosa which was published in Bologna in 1659.

One of the most important articles in the book is the bibliographical chapter by Professor R. A. Sampson on the books exhibited at the time of the meeting. Professor Sampson has given a careful description of each of the early editions, with numerous facsimiles, including two pages from the Aritmetische vnd Geometrische Progress Tabulen of Jobst Buergi, printed in Prague in 1620. These facsimiles show the title page and also one page of the tables. Since this was lent by the Town library of Dantzig, and was to have been returned early in August, 1914, it would be interesting to know where it has been stranded during the last two and a half years.

The mathematical articles include a study of Napier's rules, by Professor Somerville; a consideration of fundamental trigonometric and logarithmic tables, by Professor Andoyer; a description of the work of Edward Sang, by Professor Knott; a study of formulas used in calculation for the development of a function of two variables in spherical harmonics, by Professor Bauschinger; a study of nomograms and of the early calculating machines, by Professor d'Ocagne; a new table of natural sines, by Mrs. Gifford; a consideration of the arrangement of tables, by Dr. J. R. Milne; a note on 'critical' tables, by Mr. T. C. Hudson; a study of economy of entries in tables, by Professor J. E. A. Steggall; the graphical treatment of crystallographic problems, by Dr. A. Hutchinson; a method of computing logarithms by simple addition, by Mr. Schooling; a question as to reducing to a minimum the mean error of tables, by Mr. Erlang; a study of the extension of accuracy of tables by improvement of differences, by Dr. W. F. Sheppard, a paper of unusual interest; a method of finding antilogarithms without tables, by Dr. Artemas Martin; the approximate determination of the functions of an angle, by Mr. H. S. Gay; and a study of life probabilities on the Goldziher criterion, by Mr. Quiquet.

The work is illustrated with great care. The colored portrait of Napier, from the original painting in the University of Edinburgh, is a very welcome addition to the list of portraits of mathematicians now available for study, and the fascimiles from early printed works are of great value.

It is impossible within the limits of a review of this nature to enter into an extended description of the articles. Suffice it to say that there has never appeared a work on the history of any single mathematical discovery that has been so sumptuously published or that contains so much valuable material for the student of the history of mathematics. May we not hope that the forthcoming two-hundredth anniversary of the death of Newton may see a similar meeting in Cambridge, with the result of the publication of a similar volume relating to the life and works of the world's greatest mathematical physicist?

DAVID EUGENE SMITH.

A Course in Mathematical Analysis. By EDOUARD GOURSAT; translated by E. R. HEDRICK and OTTO DUNKEL. Volume II, Part I: *Functions of a Complex Variable.* Boston, Ginn and Company, 1916. v+259 pp.

THOSE who are familiar with Professor Hedrick's excellent translation of the first volume of this standard French treatise will welcome this continuation of his work. To have books of this type available for the use of students who do not read French with facility is a great boon to American teachers. In particular the present volume is admirably adapted for use as a text in an introductory course on the theory of functions of a complex variable.

The content of the book has already been summarized in Professor Osgood's review of the first edition of the original work.* With regard to the merits of the translation, it is sufficient to say that it maintains the high standard already set in the translation of the first volume. The separation into two parts of the translation of volume II increases the availability of the work for use in American universities.

CHARLES N. MOORE.

A Review of High-School Mathematics. By WILLIAM DAVID REEVE and RALEIGH SCHORLING. Chicago, The University of Chicago Press, 1915. x+70 pp.

THIS little handbook has now been out for nearly two years, and has had an opportunity of being put to the test. What the results of such a test are, however, can only be determined by the users. Nevertheless it is evident to anyone that such a handbook must have its chief value in a school where the course of study is the same as that followed in the exercises. The authors have evidently had this in mind, for there are not any very serious departures from the sequence of topics generally adopted in our best American schools. Even with a class which begins algebra in the conventional manner there is no reason why the slight amount of intuitive geometry in the sections devoted to the equation should not be welcome.

On the whole it may be said that the book is modeled somewhat after the exercise manuals so commonly used in Europe. In schools in which the textbooks in use do not supply sufficient review material, which is not generally the case with the leading American textbooks, this manual may be found helpful.

DAVID EUGENE SMITH.

* BULLETIN, vol. 15 (1908–09), pp. 120–126.

Plane Geometry. By JOHN W. YOUNG and ALBERT J. SCHWARTZ. New York, Henry Holt and Company, 1915. 223+x pp.

THE need of a textbook such as Young and Schwartz's Plane Geometry is perhaps best stated by the authors themselves in their preface. "As a result of the widespread discussion during recent years on the improvement of our courses in elementary geometry, the majority of thoughtful teachers appear to have reached substantial agreement on at least one point: To begin the course in plane geometry in the traditional formal manner is pedagogically irrational and scientifically unnecessary. There has accordingly arisen an increasing demand for a textbook which will supply a pedagogically rational approach to the study of plane geometry without sacrificing the logical structure of the subject."

The authors have departed from the orthodox formula for writing plane geometry texts. They do not begin with a formal (and in the eyes of the student also formidable) set of definitions and axioms followed by a proposition of the type: All right angles are equal. Instead the first chapter of 46 pages seeks to make the student familiar with the common geometric concepts. In the second chapter such notions as undefined term, fundamental proposition, geometric proof are clearly explained. Then begins the formal geometry. One finds considerable change in the list of propositions from what one ordinarily sees in an elementary text. Very many theorems which neither serve as links in the logical development nor are important in themselves have been omitted. In the space thus gained the definitions and elementary applications of the trigonometric ratios are introduced.

The authors have made geometry 'easy' in the sense that their explanations and proofs are put in such language as to be intelligible to the average student. The problems are numerous and well selected. Mechanically the book is well gotten up. The use of two color printing for the figures, the auxiliary and construction lines being in a quiet green (not bright red), helps materially. It may be that the reviewer is prejudiced since the book happens to agree with his notions of what an elementary plane geometry text should contain, and it is for that reason that he believes that it is a most welcome and valuable addition to our small list of *good* mathematical textbooks.

M. G. GABA.

Ruler and Compasses. By HILDA P. HUDSON, M.A., Sc.D. London, Longmans, Green and Company, 1916. 143 pp. 93 figures. Price $1.80.

THIS little book is a welcome addition to the literature on the boundary between elementary and advanced mathematics. While a number of excellent texts exist in other languages, either originally or by translation, heretofore we have had neither in English. The present claims only to be a compilation from other books and memoirs on the subject, yet it is really much more, as it is full of the ingenious devices peculiar to the British school of mathematics.

With the exception of the well known book of Enriques* it is more systematic than any of the preceding treatises, and excells that one by being more practical and less verbose. In fact, in places the book is too compact to be of maximum service.

The problem is introduced by comparing the postulates and constructions of Euclid and showing that every construction is based on the interpretation of the solution of linear and quadratic equations. This excellent presentation is followed by a survey of the methods by which the various problems are to be attacked; it is unfortunate to bring in a considerable number of new words here, the meaning of which is given later.

The elements of cartesian geometry are developed and applied, but the explanations are too brief. A straight line is assumed tacitly to have a linear equation; the coordinates of the point common to two lines are shown to be expressible rationally in the coefficients in the equations of the lines. The section on the domain of rationality is well written.

The regular polygons are scarcely mentioned. It is stated entirely without proof that the inscription of a regular polygon depends on a binomial equation; the limitation on the exponent of the unknown is clearly derived. The entire construction of the heptadecagon is disposed of in ten lines and a figure.

The treatment of constructions by means of the ruler alone is prefaced by a discussion of projective properties, including vanishing lines, Desargues's theorem, cross ratio, harmonic section, involution, homography, and double elements. Many of these are given entirely without proof, and in no case is the

*Questioni riguardanti la geometria elementare, Bologna, 1902; German translation by Fleischer: Fragen der Elementargeometrie, Leipzig, 1907.

proof full enough to be of much use. On the other hand, the application to elementary metrical cases is much more satisfactory.

Constructions by ruler and compasses bring in the cross ratio of four points on the circle; otherwise it keeps closely in touch with principles developed by Euclid. Then follows a chapter which employs the method of three trials, the idea of displacement, similarity, inversion, and duality, followed by another devoted to special devices on account of limited space, etc.

The last two chapters are given up to constructions made by the rulet and one fixed circle, and by the compasses alone, respectively; each contains a scheme of comparison of the relative merits of different constructions.

To sum up, the immediate practical problem is everywhere well treated, but the foundations, taken from post-euclidean ideas, are by no means so well done. Many of these ideas could have been dispensed with altogether, and the others easily proved for the purpose in hand, under restrictive hypotheses. It is true that this alternative also has objections; if a reader delves further into projective and analytic geometry, he would approach the fundamental theorems under false impressions. To one already familiar with these two subjects, the whole problem becomes an easy one, but to those who are familiar with elementary geometry and algebra only, the choice of the restrictive premises seems to the reviewer to be the wiser procedure.

The style is clear and the figures are well drawn; one unusual feature is that every algebraic equation is written in boldfaced type, making a formidable-looking page. The book is not provided with an index, nor are more than a very few exercises left for the reader to work out.

VIRGIL SNYDER.

Calculus. By HERMAN W. MARCH and HENRY C. WOLFF. (Modern Mathematical Texts, edited by C. S. SLICHTER.) New York, McGraw-Hill Book Company, 1917. 16+360 pp. $5 \times 7\frac{1}{2}$. 125 figs. Price $2.00.

HAVING in mind particularly the needs of technical students, the authors have used for illustrative purposes many practical problems. For the same reason the exercises contain many problems drawn from engineering. The book contains

about the average number of applications. There are other books on the calculus which contain many more applications than the present volume. Two problems are discussed in some detail which are omitted in most calculus books, viz., the cable with a uniform horizontal load (parabolic cable) and the cable with equal loads on equal lengths (catenary).

The authors have exercised considerable care and have met with more than ordinary success in making clear the meaning of an infinitesimal, curvature, and mean value. Duhamel's theorem is explained with unusual clearness and is used consistently in obtaining integral forms. Illustrative examples are numerous.

The preface states that a large number of drill problems have been inserted. In general we find this to be true. None of the lists are excessively long. Some are in our opinion too short. Chapters XVI and XIX end with no problems. We find other chapters ending with 1, 2, 3, and 5 problems. Answers are given to approximately fifty exercises.

The subject of integration is first brought to the student's attention on page 46 just after he has learned to differentiate the power function. Thereafter whenever he is shown how to differentiate a new function he is at once also shown how to apply the result to the integration of certain forms.

In the short sixteen-page chapter on differential equations no use is made of the initial conditions to determine the constants of integration with the exception of the discussion of a damped harmonic motion in the last article of the book. No exercises are given after this discussion. The subjects of center of gravity and moments of inertia have been treated somewhat more fully than is usual.

The book is designed for a course of four hours a week throughout the college year. But it is easy to adapt it to a three-hour course by suitable omissions. On the whole the book is a good one and has added its share to the general endeavor to obtain better texts.

W. V. Lovitt.

Die Differentialgleichungen des Ingenieurs. By Dr. W. Hort, Engineer in the Siemens-Schuckert Works. Berlin, Springer, 1914. 540 pp.

This volume cannot fail to be of interest to teachers of mathematics in American schools of engineering. The follow-

ing paragraphs taken from the preface indicate the controlling purpose of the author.

"Da ich möglichst alle für Aufgaben des Ingenieurswesen wichtigen Methoden bringen wollte, habe ich mich veranlasst gesehen, die Reihenentwickelungen nach Frobenius nebst der damit in Zusammenhang stehenden Ermittelung der logarithmenbehafteten Integrale linearer Differentialgleichungen zu behandeln. Bekanntlich werden diese Verfahren in der Behältertheorie gebraucht. Daneben werden einige Fragen, die mit der Technik nicht in unmittelbarem Zusammenhang stehen, wie z. B. die Integration der Differentialgleichungen der Planetenbewegung, ihres allgemeinen Interesses halber erörtert." * * * "Im Interesse der Anwendungen sind auch die Differenzengleichungen wenigstens in einem kurzen Abriss aufgenommen werden."

Then comes the significant statement:

"Beim Abschluss des Druckes ersehe ich aus Nr. 20 der *Zeitung des Vereines deutscher Ingenieure*, dass der deutsche Ausschuss für technisches Schulwesen in seinem fünften Bericht den gleichen Anschauungen Ausdruck gibt, die mir den Abstoss zur Abfassung dieses Buches gegeben haben."

For the reviewer the natural query as to what the book stands for in engineering education in Germany is answered in the paragraph just quoted. One is at first inclined to conclude that the author's point of view is a personal one. In fact, the scope of the book is so comprehensive,—including in the first part (pages 1–253) a thorough course on ordinary differential equations, with applications to a great variety of engineering problems, and in the second part (pages 254–519) an exposition of the differential equations of mathematical physics, with emphasis on such problems as Helmholtz's vortex theory and Lorenz's investigations on turbines,—that the first thought is that the author takes an extremely advanced position as to the possible use a practising engineer may have to make of mathematics. But the authority of the committee on technical education referred to puts the matter in a different light. Evidently the position that engineering students should be trained in mathematics far beyond the calculus receives organized support in Germany.

The book can be used with advantage by American teachers as a storehouse of problems with which to vitalize the subject of differential equations.

PERCEY F. SMITH.

Fundamental Sources of Efficiency. By FLETCHER DURELL. Philadelphia and London, J. B. Lippincott Company, 1914. 368 pp.

IN these latter days of accelerated national development, an era in which whole philosophies of life are being subjected to their supreme test—the right to live—the old principle of the survival of the fittest has assumed a new aspect and we now recognize it in the problem of national efficiency.

We have all read of the wonderful results obtained by men of genius through the intelligent application of a few simple principles of efficiency in concrete fields. The magazines have been flooded with stories using these as themes. Technical journals have presented arrays of facts and figures illustrating the astounding results obtained through their use; efficiency engineering is a modern and live profession; economists are dividing themselves into schools in accordance with their beliefs concerning the workings of these principles and some philosophers tell us that the social order is rapidly changing—that intelligent cooperation is superseding individualism.

We have seen manufacturing plants reorganized; yes, even self-appointed committees of politicians from state legislatures have set themselves the—to them—simple task of reorganizing great universities.

But we have not—at least not all of us—read a book such as that under review, in which are summarized and analyzed into a few elemental principles the various forms and sources of efficiency. It is so written that the general reader may enjoy its pages and a mathematician may easily find in the groups of exercises at the close of the chapters many a mental stimulus. An efficiency expert would find the summaries near the ends of the chapters logical recapitulations of discussions carried out—bird's-eye views of the fields covered therein.

The following are the headings of a few of the 18 chapters and appendices *A* and *B* contained in the book: II. Reuse, III. The unit and its multiplier, IV. The group, · · ·, IX. Expenditure and results, · · ·, XIII. Rhythm. These titles are sufficient to suggest, perhaps outline, the nature of the discussion in each chapter. The principles and sources of efficiency are discussed in detail and a wealth of concrete illustrations cited. Among these are many of the classic applications of which we have read and which have shown in their

simplicity and results that the whole subject is well worth while.

Any instructor might easily learn many a lesson of value from the pages of the book—if he cared to investigate, for example, just how efficiently he is handling a section of students in any subject he may be teaching.

In Chapter XVII the author shows how the various efficiency methods advocated apply to such specimen departments of human thought and endeavor as psychology, education, sociology, business, etc. A brief historical survey is given in Appendix *B*.

ERNEST W. PONZER.

NOTES.

AT the meeting of the London mathematical society on February 1 the following papers were read: By P. A. MACMAHON: "The significance of a certain algebraic function in the theory of distributions" and "The number of ways of pairing off the members of two identical sets of different quantities"; by W. H. SALMON: "Curves of constant torsion."

AT the meeting of the Edinburgh mathematical society on March 9 the following papers were read: By G. B. JEFFERY: "The transformations of axes for Whittaker's solution of Laplace's equation"; by H. DATTA: "On the failure of Heiermann's theorem."

AMONG recent mathematical works published by Gauthier-Villars are the following: Œuvres de Cauchy, 2e série, Tome XII (Nouveaux Exercices d'Analyse et de Physique, Tome 2), Œuvres de Poincaré, Tome II, and two volumes of the Borel series of monographs, Leçons sur les Méthodes de Sturm dans la Théorie des Equations différentielles linéaires et leurs Développements modernes, by Professor MAXIME BÔCHER, and Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire, by Professor C. DE LA VALLÉE POUSSIN. Tome XIII, 2e série (Nouveaux Exercices d'Analyse et de Physique, Tome 3), of the Œuvres de Cauchy is now open to subscription, and Tome I of the Œuvres de Poincaré is in press.

THE Rice Institute, of Houston, Texas, has just issued, in three volumes, the Book of the Opening of the Rice Institute, an account of the academic festival held in celebration of its formal opening in October, 1912. These volumes contain, in addition to an account of the exercises of dedication, the inaugural lectures delivered by the delegates of various institutions. Among these are the following mathematical lectures: "Molecular theories and mathematics," "Aggregates of zero measure," and "Monogenic uniform non-analytic functions—the theories of Cauchy, Weierstrass, and Riemann," by Professor EMILE BOREL, delegate of the University of Paris; and "Henri Poincaré," "The generalization of analytic functions," and "On the theory of waves and Green's method," by Professor VITO VOLTERRA, delegate of the University of Rome.

BY recent action of the Board of Trustees of the University of Chicago, the President of the University, on recommendation of the head of a department, will welcome doctors of philosophy of the University of Chicago and other universities as guests of the university, with the privilege of attending seminars and of carrying on research in the laboratories and libraries. There will be no charge except for laboratory supplies and a nominal laboratory fee where laboratory work is done. Arrangements should be made with the university in advance.

THE Adams prize of Cambridge University has been awarded to J. H. JEANS, former fellow of Trinity College, for his essay: "Some problems of cosmogony and stellar dynamics." The value of the prize is £250.

PROFESSOR A. N. WHITEHEAD has been elected president of the British mathematical association.

MR. P. A. MACMAHON has been elected president of the Royal astronomical society.

PROFESSOR EDWARD KASNER has been elected a member of the National Academy of Sciences.

DR. E. A. T. KIRCHER, of Harvard University, has been appointed instructor in mathematics at the University of Minnesota.

MR. T. R. HOLLCROFT has been appointed instructor in mathematics at Columbia University.

MR. W. H. WILSON, of the University of Illinois, has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

PROFESSOR GASTON DARBOUX, of the University of Paris, and permanent secretary of the Paris Academy of Sciences, died February 22, at the age of 74 years. He was author of a four-volume treatise on differential geometry bearing the title Théorie générale des surfaces, a work on orthogonal systems of surfaces, one on the cyclide and allied surfaces, and a large number of memoirs in the leading mathematical periodicals. During the last thirty years he has been a member of more than fifty learned societies.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAILEY (F. H.). See WOODS (F. S.).

BARROW (I.). Geometrical lectures. Translated with notes and proofs by J. M. Child. London, Open Court, 1917. 8vo. 14+218 pp. 4s. 6d.

BÔCHER (M.). Leçons sur les méthodes de Sturm dans la théorie des équations différentielles linéaires et leurs développements modernes, professées à la Sorbonne en 1913–1914. (Collection de Monographies sur la Théorie de Fonctions publiée sous la Direction de M. Émile Borel.). Recueillies et rédigées par G. Julia. Paris, Gauthier-Villars, 1917. 8vo. 6+118 pp. Fr. 5.00

BOREL (E.). See BÔCHER (M.).

CAJORI (F.). A history of elementary mathematics with hints on methods of teaching. Revised and enlarged edition. New York, Macmillan, 1917. 8vo. 8+324 pp. $1.75

CHILD (J. M.). See BARROW (I.).

JULIA (G.). BÔCHER (M.)

DE MONTESSUS DE BALLORE (R.). Leçons sur les fonctions elliptiques en vue de leurs applications. Paris, Gauthier-Villars, 1917. 8vo. 10+268 pp. Fr. 12.00

REY PASTOR (J.). Fundamentos de la geometría proyectiva superior. (Junta para ampliación de estudios e investigaciones científicas, Laboratorio y seminario matemático publicaciones, Tomo 1.) Madrid, 1916. 8vo. 22+444 pp. Pes. 12.00

SIBIRANI (F.). Riassunto formulario di geometria analytica, algebra, calcolo infinitesimale, calcolo vettoriale e meccanica razionale. Roma, Atheneum, 1915. 8vo. 4+575 pp. L. 12.00

WOODS (F. S.) and BAILEY (F. H.). Analytic geometry and calculus. Boston, Ginn, 1917. 8vo. 12+516 pp. $3.00

II. ELEMENTARY MATHEMATICS.

CAJORI (F.) and ODELL (L. R.). Elementary algebra, second year course. New York, Macmillan, 1916. 9+197 pp. $0.75

DARNELL (A.). See LYMAN (E. A.).

KLAPPER (P.). The teaching of arithmetic. New York, Appleton, 1916. 387 pp. $1.45

LYMAN (E. A.) and DARNELL (A.). Elementary algebra. New York, American Book Company, 1917. 8vo. 8+503 pp.

MILNE (R. M.). Mathematical papers for admission into the Royal military academy and the Royal military college, September–November, 1916. London, Macmillan, 1917. 32 pp. 1s.

ODELL (L. R.). See CAJORI (F.).

III. APPLIED MATHEMATICS.

AKIMOFF (N. W.). Elementary course in Lagrange's equations and their applications to solutions of problems of dynamics with numerous examples. Philadelphia, Philadelphia Book Company, 1917. $2.00

BARNARD (R. J.). Elementary dynamics of the particle and rigid body. London, Macmillan, 1917. 7+374 pp. 6s.

BRAGDON (C.). Projective ornament. Rochester, N. Y., Manas Press, 1915. 79 pp. $1.50

CHOW (H. K.). See HUNSAKER (J. C.).

CLARK (V. E.). See HUNSAKER (J. C.).

DOUGLAS (D. W.). See HUNSAKER (J. C.).

F. G. M. Manuel de mécanique conforme au programme de 1905 et de 1911. Tours, A. Mame, et Paris, J. de Gigord, 1916. 16mo. 16+432 pp.

FURUHJELM (R.). Recherches sur les mouvements propres des étoiles dans la zone photographique de Helsingfors. Helsingfors, Société de Littérature Finnoise, 1916. 190 pp.

GIBSON (G. E.). See SACKUR (O.).

HUFF (T. H.). See HUNSAKER (J. C.).

HUNSAKER (J. C.), HUFF (T. H.), DOUGLAS (D. W.), CHOW (H. K.) and CLARK (V. E.). Dynamical stability of aeroplanes. Washington, Smithsonian Institution, 1916. 78 pp.

LANCHESTER (F. W.). Flying machines from an engineering stand-point. London, Constable, 1917. 4s. 6d.

McGegan (J. E.). Star identifier and diagrams for the graphical solution of problems in nautical astronomy. London, London Name Plate Manufacturing Company and J. D. Potter, 1916. 10s. 6d.

Maillet (E.). Cours de mécanique professé à l'école des ponts et chaussées. Paris, Hermann, 1916. 8vo.

Mellor (T.) and Pearson (H. H.). Housecraft arithmetic. (Longmans' Housecraft series for secondary schools.) London, Longmans, 1917. Cr. 8vo. 112 pp. $0.45; with answers, $0.60.

Mills (C. N.). A short course in elementary mechanics. New York, Van Nostrand, 1916. 11+127 pp. $1.00

Pearson (H. H.). See Mellor (T.).

Polak (S.) and Quilter (H. C.). The teaching of drawing. Philadelphia, Lippincott, 1916. 168 pp.

Quilter (H. C.). See Polak (S.).

Riach (M. A. S.). Air screws: an introduction to the aerofoil theory of screw propulsion. London, Crosby Lockwood, 1916. 8+128 pp. 10s. 6d.

Sackur (O.). A text-book of thermochemistry and thermodynamics. Translated and revised by G. E. Gibson. London, Macmillan, 1917. 16+439 pp. 12s.

Sparagen (W.). See Wynne (W. E.).

Wynne (W. E.) and Sparagen (W.). Handbook of engineering mathematics. New York, Van Nostrand, 1917. $2.00